XXVII. *On the Rigidity of the Earth.* *By* W. Thomson, *LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow.*

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1. That the earth cannot, as many geologists suppose, be a liquid mass enclosed in only a thin shell of solidified matter, is demonstrated by the phenomena of precession and nutation. Mr. Hopkins*, to whom is due the grand idea of thus learning the physical condition of the interior from phenomena of rotatory motion presented by the surface, applied mathematical analysis to investigate the rotation of rigid ellipsoidal shells enclosing liquids, and arrived at the conclusion that the solid crust of the earth must be not less than 800 or 1000 miles thick. Although the mathematical part of the investigation might be objected to, I have not been able to perceive any force in the arguments by which this conclusion has been controverted, and I am happy to find my opinion in this respect confirmed by so eminent an authority as Archdeacon Pratt†.

2. It has always appeared to me, indeed, that Mr. Hopkins might have pressed his argument further, and have concluded that no continuous liquid vesicle at all approaching to the dimensions of a spheroid 6000 miles in diameter can possibly exist in the earth's interior without rendering the phenomena of precession and nutation very sensibly different from what they are.

3. Considerations regarding the velocities of long waves in deep sea, of tidal waves and of earthquake waves, and the harmonic vibrations of a liquid globe, having recently led me to think of the relative values of gravitation and elasticity in giving rigidity to the earth's figure, I was surprised to find that the former would have a larger share in this effect than the latter, unless the average substance of the earth had a very high degree of rigidity. For instance, I found that a homogeneous incompressible liquid globe of the same density as the mean density of the earth, if changed to a spheroidal form and then left free, influenced only by mutual gravitation of its parts, would perform simple harmonic vibrations in $47^m\ 12^s$ half-period‡. A steel globe of the same dimensions, without mutual gravitation of its parts, could scarcely oscillate so rapidly, since the velocity of plane waves of distortion in steel is only about 10,140 feet per second, at which rate a space equal to the earth's diameter would not be travelled in less than $1^h\ 8^m\ 40^s$.

4. Hence it is obvious that, unless the average substance of the earth is more rigid than steel, its figure must yield to the distorting forces of the moon and sun, not incomparably less than it would if it were fluid. To illustrate this conclusion, I have investi-

* Philosophical Transactions, years 1839, 1840, 1842.

† Figure of the Earth, edit. 1860, § 85.

‡ This will be demonstrated in a mathematical paper which the author hopes soon to communicate to the Royal Society. See §§ 55–58 of "Dynamical Problems, &c." following the present paper in the Transactions.

gated the deformation experienced by a homogeneous elastic spheroid under the influence of any arbitrarily given disturbing forces*. I thus find that if $2h'$ denote the difference between the longest and shortest diameters of the tidal spheroid, calculated on the supposition that the substance is of homogeneous (and therefore incompressible) fluid, and $2h$ the difference between the longest and shortest diameters of the spheroid into which the same mass, if of homogeneous incompressible solid matter, would be deformed from a naturally spherical figure when exposed to the same lunar or solar disturbing influence, we have (see § 34, Appendix to this paper)

$$h=\frac{h'}{1+\frac{19}{2}\frac{n}{gwr}},$$

where w denotes the mass of unit volume, and n the "rigidity" of the substance (see § 71 of the paper following the present in the Transactions); and g denotes the force of gravity on a unit of mass at the surface, and r the radius of the globe.

5. The density of iron or steel (7·8 times that of water) does not differ very much from the mean density of the earth (5·6 times that of water according to CAVENDISH'S experiment, or 6·6 according to the Astronomer Royal's). The rigidity of iron, according to experiments of my brother, Professor JAMES THOMSON†, is 10,800,000 lbs. per square inch. Since the weight of 1 lb. at Glasgow, where the experiment was made, is 32·2 British absolute units of force, we must multiply by 32·2 to reduce to kinetic measure as to force; and we must multiply by 144 to make the unit of area a square foot instead of a square inch. We thus find, in consistent absolute measure,

$$n=501\times10^{8},—$$

the unit of mass being 1 lb., the unit of space 1 foot, and the unit of force that force which, acting on one pound of matter during a mean solar second of time, generates a velocity of 1 foot per second. In terms of the same units we have $r=20,887,700$; $g=32{\cdot}14$, being about the average over all the earth; and for iron or steel $w=487$. Hence

$$h=\frac{h'}{1+\frac{19}{2}\cdot\frac{501\times10^{8}}{3308\times10^{8}}}=\frac{h'}{2{\cdot}44}={\cdot}41h'.$$

Of glass, the rigidity is, according to WERTHEIM, about one-fifth of the value we have just used as that of iron; and therefore if the earth were homogeneous of its actual mean density, and had throughout the same rigidity as that of glass, the result would be $h={\cdot}78h'$.

6. Hence it appears that if the rigidity of the earth, on the whole, were only as much as that of steel or iron, the earth as a whole would yield about two-fifths as much to the tide-producing influences of the sun and moon as it would if it had no rigidity at all; and it would yield by more than three-fourths of the fluid yielding, if its rigidity were no more than that of glass.

* The solution of this problem will be found in the paper referred to above (see §§ 47, 48).

† Cambridge and Dublin Mathematical Journal, 1848.

7. Such a deformation as this would be quite undiscoverable by any direct geodetical or astronomical observations; but if it existed, it would largely influence the actual phenomena of the tides and of precession and nutation.

§§ 8–20. *Effect of the Earth's Elastic Yielding on the Tides.*

8. To find the effect of the earth's elastic yielding on the tides, let 2H denote the difference between the greatest and least diameters of the spheroidal surface perpendicular to the resultant of the lunar or solar disturbing force*, and terrestrial gravitation supposed perfectly symmetrical about the centre, then $\frac{H}{r}$ will be the ellipticity of that spheroid; and we shall call it the *ellipticity of level produced by the lunar or solar influence on a rigid earth.* It may be remarked that H is the height of high above low water in the "equilibrium tide" of an ocean of infinitely small density covering a rigid earth.

9. Let H′ denote the height of the equilibrium tide for an ocean of density $\frac{1}{N}$ of the earth's mean density, the earth being still supposed *perfectly* rigid and covered by the ocean. Then the terrestrial gravitation level will be disturbed (as is proved in the theory of the attraction of ellipsoids) from the spherical surface to the spheroidal surface of ellipticity $\frac{3}{5}\cdot\frac{1}{N}\cdot\frac{H'}{r}$, by the attraction of the ocean in its altered figure. The ellipticity of level induced by lunar or solar influence must be added to this to give the ellipticity of actual level, which is of course the ellipticity of the free equilibrium surface of the ocean, or according to our notation $\frac{H'}{r}$. Hence

$$\frac{H'}{r}=\frac{H}{r}+\frac{3}{5}\cdot\frac{1}{N}\cdot\frac{H'}{r},$$

by which we find

$$H'=\frac{H}{1-\frac{3}{5}\cdot\frac{1}{N}}.$$

For sea-water the value of N is about $\frac{1}{5\cdot5}$; and therefore

$$H'=\frac{9\cdot2}{8\cdot2}H=1\cdot12H,$$

or only 12 per cent. more than for an ocean of infinitely small density.

10. What we have denoted above by h' is the value of H′ for N=1; and therefore

$$h'=\frac{5}{2}H,$$

* This "disturbing force" is of course the resultant of the actual attraction of either body on a unit of mass in any position, and a force equal and opposite to its attraction on a unit of mass at the earth's centre.

and

$$h=\frac{5}{2}\cdot\frac{\mathrm{H}}{1+\frac{19}{2}\cdot\frac{n}{gwr}}.$$

11. Now, according to a proposition regarding the attraction of ellipsoids already used, we have $\frac{3}{5}\cdot\frac{h}{r}$ for the ellipticity in the terrestrial gravitation level produced by the ellipticity of deformation $\frac{h}{r}$ experienced in consequence of want of perfect rigidity. Hence the ellipticity of the terrestrial gravitation level, as disturbed by lunar or solar influence, is $\frac{3}{5}\cdot\frac{h}{r}+\frac{\mathrm{H}}{r}$. This will be the absolute tidal equilibrium ellipticity of an ocean of infinitely small density covering the elastic globe; but since there is a tidal ellipticity $\frac{h}{r}$ induced in the solid itself, the height from low tide to high tide of fluid relatively to solid (that is to say, the difference of depth between high water and low water) will be

$$\left(\frac{3}{5}h+\mathrm{H}\right)-h,$$

or

$$\mathrm{H}-\frac{2}{5}h;$$

or, according to the value of h just found (§ 10),

$$\frac{\frac{19}{2}\cdot\frac{n}{gwr}\mathrm{H}}{1+\frac{19}{2}\cdot\frac{n}{gwr}}.$$

12. This result expresses strictly the height of the equilibrium tide of a liquid of infinitely small density covering an elastic solid globe. It may be regarded as a better expression of the true tidal tendency on the actual ocean than the slightly different result calculated with allowance for the effect of the attraction of the altered watery figure constituting the equilibrium spheroid, and its influence on the figure of the elastic solid; since the impediments of land and the influence of the sea-bottom render the actual ocean surface altogether different from that of the equilibrium spheroid.

13. Hence the actual tidal tendency, which would be H if the earth were perfectly rigid, is in reality

$$\frac{\frac{19}{2}\cdot\frac{n}{gwr}\cdot\mathrm{H}}{1+\frac{19}{2}\cdot\frac{n}{gwr}},$$

where n denotes what we may call the *earth's tidal effective rigidity*, being the "rigidity" of a homogeneous incompressible solid globe of equal mass which, with an ocean equal and similar to the earth's, would exhibit the same tides.

14. If, for example, we give n the value for iron or steel above indicated, the formula becomes $\cdot 59\times\mathrm{H}$. The comparison between theory and observation, owing to the

extreme complexity of the circumstances, has been hitherto so imperfect that we cannot say it disproves this result; and therefore, from tidal phenomena hitherto observed, we cannot infer that the earth is more effectively rigid than steel.

15. The value of n for glass, according to WERTHEIM, is $2160000\times144\times32\cdot2$, in British absolute units; and it reduces the formula to $\frac{1}{4\frac{1}{2}}\text{H}$. Now, imperfect as the comparison between theory and observation as to the absolute height of the tides has been hitherto, it is scarcely possible to believe that the height is in reality only two-ninths of what it would be if, as has hitherto been universally assumed in tidal investigations, the earth were perfectly rigid. It seems therefore nearly certain, with no other evidence than is afforded by the tides, that the tidal effective rigidity of the earth must be greater than that of glass.

16. Any approach to a close testing of the absolute amount of the tidal influence can scarcely be expected of either of the two great Kinetic* theories—the Oceanic theory of LAPLACE, or the Channel theory of AIRY,—as applied to diurnal or semidiurnal tides; but notwithstanding the strong contempt which has been expressed by the last-mentioned naturalist † (no doubt justly as regards false applications of it) for the Equilibrium theory ‡, we may look to it confidently for good information when it is applied to test the difference between mean fortnightly variations of level at two well-chosen stations, one in a low latitude, and the other in a high latitude. (See Note at the end of this paper.)

17. The fortnightly tide ** at each pole gives high water when the moon's declination (Δ), whether north or south, is greatest, and low water when she crosses the equator; and the whole difference in level produced by it would be

$$\text{H}'\sin^2\Delta$$

if the earth were all covered with water. The mean daily level at the equator, on the same supposition, would vary by half that amount, being low water when the moon is furthest from the equator, and high when she crosses the equator. But, owing to the actual distribution of land and water, either of those variations may be diminished by

* Dynamics meaning properly the science of force, and there being precedents of the very highest kind, for instance, in DELAUNAY'S 'Mécanique Rationnelle' of 1861, and ROBISON'S 'Mechanical Philosophy' of 1804, in favour of using the term according to its proper meaning—and the modern corrupt usage, which has confined it to the branch of dynamical science in which relative motion is considered, being excessively inconvenient and vexatious,—it has been proposed to introduce the term "kinetics" to express this branch; so that dynamics may be defined simply as the "science of force," and divided into the two branches, Statics and Kinetics. The introduction of this new term, derived from κίνησις, *motion*, or act of moving, does not interfere with AMPÈRE'S term, now universally accepted, "kinematics" (from κίνημα), the *science of movements*.

† "Naturalist. A person well versed in Natural Philosophy."—JOHNSON'S Dictionary. Armed with this authority, chemists, electricians, astronomers, and mathematicians may surely claim to be admitted along with merely descriptive investigators of nature to the honourable and convenient title of Naturalist, and refuse to accept so un-English, unpleasing, and meaningless a variation from old usage as "physicist."

‡ Encyclopædia Metropolitana, "Tides and Waves," §§ 64, 539, &c.

** AIRY'S 'Tides and Waves,' § 45.

an amount which it is impossible to estimate theoretically; but then the other must be increased by nearly the same amount. And if a denote the mean height of the sea-level above a fixed mark at the earth's north pole, about times when the moon's declination is greatest, b the corresponding mean of observations about times when she is crossing the equator, a' and b' corresponding means derived from observation at an equatorial station, and ${}_{\prime}\mathrm{H}$ something intermediate between H and H′, we must have

$$a-b+b'-a'=\tfrac{3}{2}{}_{\prime}\mathrm{H}\sin^2\Delta,$$

whatever be the distribution of land and water over the earth, only provided the fortnightly tide follows sensibly the equilibrium law, which, for moderately well-chosen stations, we may suppose it must do.

18. If, instead of being at a pole and at the equator, the stations are in latitudes respectively l and l', we should have

$$a-b+b'-a'=\tfrac{3}{2}{}_{\prime}\mathrm{H}\sin^2\Delta(\sin^2 l-\sin^2 l').$$

Now if we suppose the moon's mass to be $\frac{1}{75}$ of the earth's, we have H=1·92 foot. As H′=1·12 H, and as there is more area of water than of land over the earth, we cannot be far wrong in taking ${}_{\prime}\mathrm{H}=1{\cdot}08\,\mathrm{H}=2{\cdot}04$ feet.

The greatest value of Δ is 28° 37′; and hence, in the most favourable lunations,

$$a-b+b'-a'=\cdot 713 \text{ foot}\times(\sin^2 l-\sin^2 l').$$

19. Iceland and Teneriffe, in nearly the same longitude, and in latitudes 63° 20′ and 28° 30′, would probably be very favourable stations. For them $\sin^2 l-\sin^2 l'=\cdot 571$; and therefore

$$a-b+b'-a'=0{\cdot}407 \text{ foot},$$

or about 4·9 inches.

It is probable that carefully made and reduced observations, with proper allowance for barometric disturbances, at two such stations, would not only detect this tide, but would give a tolerably accurate determination of its amount.

20. It would be, for Iceland and Teneriffe, as found above, 4·9 inches if the earth were perfectly rigid; or 3 inches if the tidal effective rigidity is only that of steel; or about an inch if the tidal effective rigidity is only that of glass.

There seems no more hopeful way to ascertain how rigid the earth really is, than to make careful observations with a view to determining the fortnightly tide with all possible accuracy. It is possible also that very accurate observations on the semi-diurnal tides in a deep inland lake of great extent, or at distant points of the Mediterranean sea-board with only deep water intervening, might help to solve this question.

§§ 21–32. *Effects of Elastic Yielding on Precession and Nutation.*

21. If we suppose the sun, the moon, and the earth's centre to be reduced to rest at any moment, the two former to be held fast in their places, and their attractions on the latter to be balanced by an infinitely great mass held at the proper infinitely great distance in the proper direction; the tide-generating distribution of force, and the couple

tending to turn the earth round an equatorial diameter, by which precession and nutation are produced, would be left precisely as they are.

22. If the sun and moon be carried round the earth at rest, according to their true relative orbital motions, and if the infinitely distant mass be shifted continually so as always to balance their attractions on the earth's centre of gravity, all the phenomena of tides, of nutation, and of precession, will take place precisely as they do in reality.

23. If, now, merely as an artifice to avoid mathematical calculations, we suppose the earth's rotation to be stopped, and instead a repulsive force, from an infinite fixed line coinciding with the earth's axis in the first place, to be introduced—the force to vary directly as the distance from this fixed line, and to amount to $\frac{1}{289}$ of gravity at the equator,—the figure of the earth will remain unchanged.

24. Let us first suppose the earth to be either perfectly fluid, or to be a perfectly elastic homogeneous and isotropic solid, of spherical figure when undisturbed, and the sun and moon to be held fast in any positions. It will clearly remain in perfect equilibrium in the circumstances defined in §§ 21, 23, whether the sun and moon are both in the plane of its equator or not; because if it is not in equilibrium it must commence rotating with a continually accelerated motion about some axis; the fixed repelling line not being supposed to impede its motion, but merely to continue repelling according to the stated law.

25. But either sun or moon, if not in the plane of the equator, and the corresponding part of the infinitely distant balancing attractor, will exercise a couple upon the earth, by attracting the near protuberant equatorial parts more, and the remote less, than the centre. How then is this couple balanced? Clearly it is by the repulsion of the fixed repelling line on the tidal deformation produced by the sun or moon, as the case may be. Considering for simplicity only one, the sun for instance, we perceive that the equilibrium tide will produce a small elliptic deviation, superimposed on the great polar and equatorial ellipticity, the longer axis of this smaller superimposed ellipticity being in the line through the sun. Now without this the spheroid of revolution would experience no resultant action from the repelling line; but with it the actual resultant spheroid will experience, from the repelling line, a couple tending to carry away from this line the longer axis of the superimposed ellipticity. This is exactly opposite to the couple produced by the attraction of the sun; and as there can be no resultant action, we see that the equilibrium is maintained by the balancing of these two couples.

26. Hence in reality we conclude that the couple due to a disturbing body in any position attracting a rotating fluid spheroid, or elastic isotropic body naturally spherical and rendered oblate by rotation, is balanced by the couple of centrifugal force on the crowns of the tidal elongation produced by the disturbing body, provided the rotation is not so fast as to render the tidal deformation sensibly different from what it would be if the disturbing body rotated with the same angular velocity as the spheroid. This condition will, it is intended, be shown, in a subsequent communication to the Royal Society, to be essentially fulfilled when the rotation is slow enough to allow the first approximation to be used as in the ordinary investigations regarding the figure of the earth.

27. Let us now suppose the earth to be an elastic spheroid of nearly on the whole the

same figure in all its surfaces of equal density as a rotating fluid, and therefore on the whole nearly free from distorting stress in its interior. If under the lunar and solar influences it were to yield and experience nearly as much tidal elongation as it would if quite fluid, the couple to which precession and nutation are due would be very nearly balanced by the centrifugal force on the crowns of the tidal elongation, and consequently precession and nutation would be very much less than if the earth were perfectly rigid. For instance, if the earth were a homogeneous incompressible elastic solid of the same rigidity as glass, it would, as stated above, experience seven-ninths of the tidal deformation of a fluid globe of the same density. Hence seven-ninths of the couple would be balanced, and precession and nutation would be reduced to two-ninths by the elastic yielding. Even if the rigidity were as much as that of steel, the precession and nutation would not be more than three-fifths of their full amount for a perfectly rigid spheroid.

28. The close agreement between the observed amounts of precession and nutation, and the results of theory on the hypothesis of perfect rigidity, renders it impossible to believe that there is enough of elastic yielding to influence the phenomena to any considerable extent. It is worthy of remark, however, that in general the theoretical estimates of the amount of precession have been somewhat above the true amount demonstrated by observation. It seems not altogether improbable that this discrepance is genuine, and is to be explained by some small amount of deformation experienced by the solid parts of the earth, under lunar and solar influence.

29. But the only possible ground on which it could be maintained that the earth as a whole is less rigid than a solid steel globe of the same dimensions, is to assume that there is an enormous liquid vesicle, or a solid nucleus separated by a fluid layer from the outer crust, in the interior, and that the loss of precessional effective moment of inertia, owing to this portion not being carried round in the precessional movement, is almost exactly compensated by a diminution of the generating couple in very nearly the same proportion by elastic yielding. Although, considering HALLEY'S theory of the secular variation of terrestrial magnetism, and the general accordance of its results with the actual phenomena as demonstrated by the best observations made up to the present time*, it would be most rash to say that it is very improbable there is a solid iron nucleus sunk to the centre of a hollow central vesicle of fluid in the earth, yet it seems to me excessively improbable that the defect of moment of inertia due to fluidity in the earth's interior bears approximately the same ratio to the whole moment of inertia, as the actual elastic yielding bears to the yielding which would take place if the earth were perfectly fluid. Avoiding conjectural assumptions, however, I conclude that either this proportion is approximately fulfilled, or both the following propositions are true:—

I. The defect of moment of inertia, owing to fluidity in the interior, is small in comparison with the whole moment of inertia of the earth.

II. The deformation experienced by the earth, owing to lunar and solar influence, is small in comparison with what it would experience if it were perfectly fluid.

* General SABINE, Proceedings of the Royal Society, 1862.

30. It is easily seen that the first of these propositions is not opposed to HALLEY'S theory. For instance, if there were a spheroidal iron core* 2000 miles diameter, cool and magnetic to within 100 miles of its surface, sunk to the centre of a spheroidal space of lighter fluid 3000 miles diameter, enclosed within a solid crust 2500 miles thick, the moment of inertia would be only about one-half per cent. less than it would be if the whole were rigid. Far less magnetism than such a nucleus could retain would be sufficient in amount to account for either the whole of "terrestrial magnetism" as manifested at the surface, or for the secular variations of it which have been observed. Whatever may be thought of the probability of this hypothesis, the barest possibility that it may be true renders it an interesting problem for mathematicians to find the precessional movement of a rigid spheroid sunk in a lighter liquid enclosed within a rigid spheroidal shell. A solution, founded on the supposition that all the bounding surfaces are truly elliptic and of small ellipticity, and that the fluid portion is of uniform density, is to be obtained with ease in an extremely simple form, and might be useful as a guide for speculation.

31. That the "tidal effective rigidity" (§ 13), and what we may similarly call the "precessional effective rigidity" of the earth, may be both several times as much as that of iron (which would make the phenomena both of tides and of precession and nutation sensibly the same as if the earth were perfectly rigid), it is enough that the actual rigidity should be several times as great as the actual rigidity of iron, throughout 2000 or more miles' thickness of crust.

32. At the surface, and for many miles below the surface, the rigidity is certainly very much less than that of iron (how much less might be estimated if we had trustworthy data as to the velocity of natural or artificial earthquake waves through short distances); and therefore at great depths the rigidity must be enormously greater than at the surface. That both the rigidity and the resistance to compression should be much greater several hundred miles down than at the surface, seems a natural consequence of the enormous pressure experienced at those great depths by the matter of the earth.

Note on the Fortnightly Tide.

33. In water 10,000 feet deep (which is considerably less than the general depth of the Atlantic, as demonstrated by the many soundings taken within the last few years, especially those along the whole line of the Atlantic telegraph cable, from Valencia to Newfoundland) the velocity of long free waves is 567 feet per second†. At this rate the time of advancing through 57° (or a distance equal to the earth's radius) would be only ten hours. Hence it may be presumed that, at least at all islands of the Atlantic, the fortnightly tide should follow sensibly the equilibrium law.

* An ancient cold iron meteorite which may have entered a nebula of smaller bodies and formed the nucleus of our present earth, which under such circumstances could not but be built up and heated by attracting them to itself.

† AIRY, § 170.

"In the Philosophical Transactions, 1839, p. 157, Mr. WHEWELL shows that the "observations of high and low water at Plymouth give a mean height of water increasing "as the moon's declination increases, and amounting to three inches when the moon's "declination is 25°. This is the same direction as that corresponding in the expression "above to a high latitude. The effect of the sun's declination is not investigated from "the observations. In the Philosophical Transactions, 1840, p. 163, Mr. WHEWELL has "given the observations of some most extraordinary tides at Petropaulofsk in Kam-"schatka, and at Novo-Arkhangelsk in the island of Sitkhi on the west coast of North "America. From the curves in the Philosophical Transactions, as well as from the "remaining curves relating to the same places (which, by Mr. WHEWELL's kindness, "we have inspected), there appears to be no doubt that the mean level of the water at "Petropaulofsk and Novo-Arkhangelsk rises as the moon's declination increases. We "have no further information on this point."—AIRY's 'Tides and Waves,' § 533.

APPENDIX, added January 2, 1864.

34. Let the difference of longest and shortest radii, which would be produced by lunar and solar influence in the two cases—of the earth supposed a homogeneous incompressible fluid tending to the spherical shape by gravitation alone, and supposed a homogeneous incompressible elastic solid without mutual gravitation but tending in virtue of its elasticity to the spherical figure—be denoted by h' and h'' respectively; and let h be the difference of greatest and least radii when both gravity and elasticity act jointly to maintain the spherical figure. We shall have obviously

$$\frac{1}{h}=\frac{1}{h'}+\frac{1}{h''}.$$

For the distorting force, being balanced by elasticity and by gravity jointly, may be divided into two parts, one $\frac{h}{h''}$ of the whole, balanced by elasticity alone, and the other $\frac{h}{h'}$, balanced by gravity alone; and therefore $\frac{h}{h''}+\frac{h}{h'}=1$.

But, by § 53 of the following mathematical investigation regarding elastic spheroids, we have $h''=\frac{3}{2}\frac{\mathrm{m}}{c^3}\frac{5w}{19n}r^3$, where m denotes the mass of the disturbing body, and c its distance from the earth's centre. With the same notation we have, by the aid of § 51 of the same paper, $\mathrm{H}=\frac{3}{2}\frac{\mathrm{m}}{c^3}\frac{r^2}{g}$, where H has the meaning defined above in § 8 of the present paper; and therefore, § 10, $h'=\frac{5}{2}\cdot\frac{3}{2}\frac{\mathrm{m}}{c^3}\frac{r^2}{g}$. From this and the value above for h'', we have $\frac{h'}{h''}=\frac{19n}{2gwr}$, and, as we have just seen that $h=\frac{h'}{1+\frac{h'}{h''}}$, we have the result used in § 4.